

(Loop) quantum gravity and the inflationary scenario*

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Abstract

Quantum gravity, as a fundamental theory of space-time, is expected to reveal how the universe may have started, perhaps during or before an inflationary epoch. It may then leave a potentially observable (but probably minuscule) trace in cosmic large-scale structures that seem to match well with predictions of inflation models. A systematic quest to derive such tiny effects using one approach, loop quantum gravity, has, however, led to unexpected obstacles. Such models remain incomplete, and it is not clear whether loop quantum gravity can be consistent as a full theory. But some surprising effects appear to be generic and would drastically alter our understanding of space-time at large density. These new high-curvature phenomena are a consequence of a widening gap between quantum gravity and ordinary quantum-field theory on a background.

How much of quantum gravity can be tested by cosmological observations? This question is non-trivial, not only because the understanding of “quantum gravity” depends on the approach one takes. A second, and perhaps more important, difficulty consists in the fact that several key aspects of physical properties usually associated with theories of gravity play interconnected roles in this context. Based on the lessons of general relativity, a quantum theory of gravity, in general terms, is expected to entail (i) a modified dynamics, given by a gravitational force with quantum corrections; (ii) new quantized modes, gravitons; and (iii) a quantum version of space-time structure.

While the first two aspects are commonly exploited for proposals of potentially observable consequences in cosmology, the last one is most crucial at a conceptual level. Only if this subtle issue is addressed can one be sure that one is testing quantum gravity, as opposed to a quantum field theory of tensor modes on some curved background: Classical gravity, described by general relativity, is characterized not only by a certain form of dynamics but also, and more fundamentally, by a large class of covariance symmetries. These symmetries characterize the structure of space-time in classical gravity as a Riemannian manifold of Lorentzian signature. Quantizing fields on such a manifold is not the same as quantizing space-time itself: The full (spatial) metric would not be subject to quantum fluctuations. A full quantization of the metric, on the other hand, requires great care

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because, in the absence of a background, it can be changed by a large class of transformations which must in some way be respected after quantization, or else the theory would be coordinate dependent and therefore meaningless.

Quantizing highly symmetric theories is always a delicate process, and the decades of unsuccessful attempts to find a complete and consistent theory of quantum gravity bear witness to the fact that this theory is no exception. Quantum field theory on a curved background does not quantize space-time and its symmetries, and therefore leaves out an important part of the question of quantum gravity. A common, and often implicit, assumption is that quantum field theory on curved space-time may be a good approximation to quantum gravity in relevant cosmological regimes. However, the process of gravitons seeding a background of tensor modes is a rather strong quantum-gravity effect; in fact, it is often taken as the best (or only) way to “observe” or “establish the quantization of gravity” [1]. If this is the case, is it legitimate to treat quantum gravity as some version of quantum field theory on a curved background, bypassing a quantum theory of curved space-time itself?

The present contribution does not provide a universal answer to this question. However, an example will be given in which quantum field theory on a curved background turns out to be not just a poor approximation to a specific model of quantum gravity, but to result in a different and incompatible scenario. The model has been formulated in the framework of loop quantum gravity, following systematic attempts to derive potential observational consequences of this theory. Loop quantum gravity remains incomplete, and it is not clear how generally the effects appear that are responsible for the mismatch between quantum space-time and quantum field theory on a curved background, even though at present they seem to be rather generic. The result is sobering and provides much caution against too-optimistic presentations of potentially observable quantum-gravity effects, not only in loop quantum gravity itself but perhaps also more generally. This article therefore refrains from presenting specific details of potential observations in loop quantum gravity, as they all appear to be largely ambiguous at present. For some of the known options, interested readers are referred to reviews such as [2, 3].

The different aspects of quantum gravity, listed above, are all interrelated. Graviton scattering implies loop corrections in the perturbative law of the gravitational force. A fundamental version of this law remains to be found, owing to the problem that gravity is not renormalizable. But in this context it is sufficient to consider an effective theory of gravity [4, 5] in which loop corrections to the gravitational force can be discussed in a well-defined way — so long as a background treatment of space-time is valid. The structure of space-time enters into the considerations because it is closely linked to the modes and dynamics of gravity. One obtains the Einstein–Hilbert action from the covariance symmetries of space-time, leaving much less freedom for interactions compared with other fields (on a given background). An effective theory of gravitons on a classical background contains the usual higher-curvature terms.

When one quantizes gravity, the structure of space-time may change, depending on the specific approach used. Well-known proposals include non-commutative geometry or discreteness. If space-time is no longer modeled by a Riemannian manifold, it is not clear

how general covariance can still apply in a meaningful way, but there must be some form in any consistent approach because the breaking of covariance symmetries would amount to a gauge anomaly. Only after the form of space-time has been determined in a given approach can one proceed and compute dynamical quantum corrections. Effective theory, as the main and most powerful tool to derive reliable quantum effects, relies on knowledge of the symmetries of a theory. Symmetries determine which terms an effective action may contain, the coefficients of which one can then compute by more-detailed calculations. Non-Riemannian structures can lead to quantum-gravity effects which a treatment of quantum-field theory on a curved background of classical form would miss.

A canonical approach is often useful in order to reveal non-trivial aspects of quantum theories. It is of special importance in quantum gravity because it does not presuppose what structure space-time may have. Symmetries are represented by generators, which are quantized in a canonical approach and may have quantum corrections. If also their algebra turns out to be modified (but not broken for an anomaly-free theory), a new structure of space-time emerges. The rest of this article attempts to review these features without too many technical details, followed by an example in which quantum-field theory on a curved background differs crucially from a model of (loop) quantum gravity.

In any quantum theory, dynamics is generated by a Hamiltonian operator \hat{H} , in quantum mechanics following the Schrödinger equation $i\hbar\partial\psi/\partial t = \hat{H}\psi$. In a covariant theory, the time coordinate can be changed freely (and locally), so that \hat{H} must also be a generator of symmetries. This dual role of Hamiltonian operators is one of the reasons why quantum gravity is complicated and incomplete, but its conceptual consequences can be explored by well-known methods.

General covariance in other words is local Lorentz (or Poincaré) invariance. In contrast to a quantum-mechanics situation (or minisuperspace quantum cosmology) with a single time parameter t and its symmetry of rigid time translations $t \mapsto t + a$, a covariant theory allows one to change time independently at different positions as well as non-linearly: $t \mapsto t'(t, x)$. Correspondingly, there is not just one generator \hat{H} but a whole family, or a function $\hat{H}(t, x)$.

For transformations $t'(t, x)$ linear in t and x , one obtains rigid time translations and Lorentz boosts, as pictured in Minkowski diagrams familiar from special relativity; see Fig. 1. The commutator of a time translation and a boost is a spatial translation, and the commutator of two boosts is a spatial rotation. The whole Poincaré algebra is generated if one starts with a family $\hat{H}(t, x)$ of Hamiltonians generating linear space-time transformations.

A generally covariant theory allows one to perform not only local but also non-linear space-time transformations. The Minkowski diagrams of special relativity may then be replaced by arbitrary deformations of curved spatial slices in space-time (Fig. 1). The position-dependent generator $\hat{H}(t, x)$ is then conveniently replaced by a “time deformation” $T[N] = \int d^3x N(x) \hat{H}(t, x)$, where N is a function on the spatial slice (t constant) so that $T[N]$ signifies an infinitesimal deformation along the normal direction in space-time, by a distance $N(x)$ at point x . Similarly, spatial deformations within a slice, along vector fields \vec{w} , are denoted as $S[\vec{w}]$. (The function N and vector field \vec{w} appear as lapse and shift in a

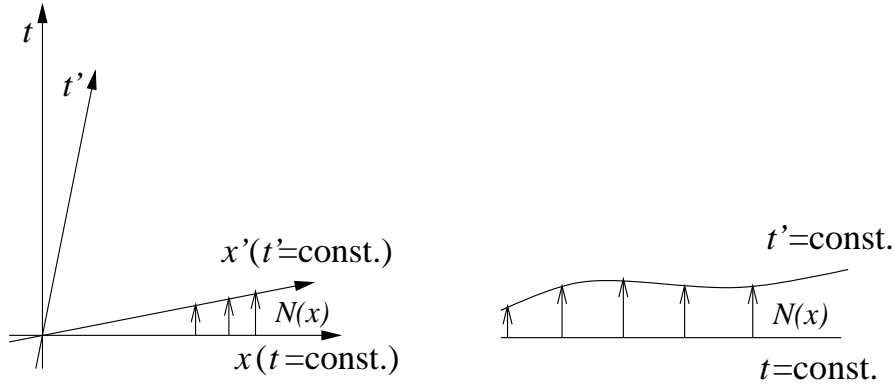


Figure 1: An infinitesimal Lorentz transformation shifts points on constant-time surfaces in a linear fashion, normal to the surface (left). A transformation of general covariance moves points normal to a constant-time surface, but non-linearly (right).

canonical formulation of the gravitational dynamics, such as the ADM decomposition [6]. See [7] for details.)

The commutators of such transformations can be derived from the geometry of spatial 3-manifolds embedded in space-time, resulting in the hypersurface-deformation algebra [8]

$$[S[\vec{w}_1], S[\vec{w}_2]] = S[\mathcal{L}_{\vec{w}_1} \vec{w}_2] \quad (1)$$

$$[T[N], S[\vec{w}]] = -T[\mathcal{L}_{\vec{w}} N] \quad (2)$$

$$[T[N_1], T[N_2]] = S[N_1 \vec{\nabla} N_2 - N_2 \vec{\nabla} N_1]. \quad (3)$$

For linear \vec{w} and N and flat (Euclidean) spatial slices in Minkowski space-time, one can confirm that the hypersurface-deformation algebra is reduced to the Poincaré algebra.

Unlike the Poincaré algebra, however, the hypersurface-deformation algebra depends on the metric induced by space-time on a spatial slice: The gradients in (3) are defined only if a metric is provided ($\nabla^a N = h^{ab} \partial N / \partial x^b$ with the inverse spatial metric h^{ab}). Even when space-time is Minkowski but the spatial slices are not flat do we have a metric entering the algebraic relations, in addition to the parameters N and \vec{w} of generators. Mathematically, the commutators do not provide a strict Lie algebra, but a Lie algebroid [9]. Physically, the presence of the inverse metric means that canonical quantum gravity, which provides operators for the spatial metric, could lead to quantum corrections in the structure functions of the hypersurface-deformation algebra. The structure of space-time may then change.

A more-detailed analysis reveals that structure functions of symmetry algebras are *not* subject to quantum corrections in a large number of cases, provided that the quantum generators can be represented without anomalies [10]. However, an exception is given by theories that quantize by not fully removing their regulators. Loop quantum gravity is of this form because it replaces connection or curvature components A_a^i by exponentiated

versions, so-called holonomies (or parallel transport)

$$h_e(A) = \mathcal{P} \exp(\int_e \tau_i A_a^i t_e^a ds) \quad (4)$$

with $SU(2)$ -generators τ_i and for any spatial curve e with tangent vector t^a (and path ordering denoted by \mathcal{P}). Since the local gauge group is compact, the matrix elements of $h_e(A)$ are bounded functions of A_a^i . Compared with a direct quantization of A_a^i in Wheeler-DeWitt fashion, holonomies have more convenient mathematical properties as operators [11, 12].

The regulator, given by extended curves used to integrate holonomies or parallel transport, can be removed without much trouble when one assumes that the theory is invariant under spatial deformations [13]. Any extended curve can then be equivalent to one arbitrarily close to a single point, and holonomies approximate curvature components exceedingly well. However, a discussion of anomaly freedom of a quantized hypersurface-deformation algebra cannot assume invariance under spatial deformations because these transformations are an important part of the algebra. The regulator is no longer being removed, and quantum corrections in the structure functions of the algebra are possible.

At this stage, an effective approach to canonical quantization and symmetries is useful [14, 15]. For decades, it has been an unsolved and intimidating problem to find quantum generators \hat{C}_I of the gravitational symmetries in (1)–(3) and to compute their commutators $[\hat{C}_I, \hat{C}_J]$. (As in any theory with local symmetries, there are infinitely many generators. The index I may therefore lie in a continuous range.) For a consistent and anomaly-free quantization, the commutators must form a closed algebra, that is $[\hat{C}_I, \hat{C}_J] = \hat{f}_{IJ}^K \hat{C}_K$ with suitable structure operators \hat{f}_{IJ}^K corresponding to (1)–(3). Classically, we have a closed algebra of Poisson brackets $\{C_I, C_J\} = f_{IJ}^K C_K$ from canonical gravity, but quantization is sensitive to factor ordering and regularization, so that closure of the operator algebra is not guaranteed. The canonical effective approach provides methods by which one can compute a “quantum Poisson bracket” defined as

$$\{C_I^{\text{eff}}, C_J^{\text{eff}}\}_Q := \frac{\langle [\hat{C}_I, \hat{C}_J] \rangle}{i\hbar}. \quad (5)$$

It is evaluated for effective constraints $C_I^{\text{eff}} := \langle \hat{C}_I \rangle$ (computed in generic states) which can be written as the classical constraints plus an infinite series of quantum corrections. The crucial observation is that a closed quantum algebra with structure operators \hat{f}_{IJ}^K implies a closed algebra of effective constraints to any order of the \hbar -expansion of quantum corrections. The effective Poisson brackets are much easier to compute than operator commutators. And while the closure of an effective algebra to some order in \hbar does not imply the existence of a closed operator algebra, a negative outcome is meaningful: If the effective algebra cannot close to some order in \hbar , it is impossible to find a closed operator version. Or if the effective algebra can close only if its structure functions have a certain form of quantum corrections, the operator algebra must be modified as well. Such algebraic statements are much more general than details of the precise dynamics computed to some

order in \hbar : Even if the effective constraints to some order do not approximate the quantum dynamics well, potential closed versions of their algebra still have implications for possible forms of the quantum algebra.

In models of loop quantum gravity, one generic result of this form has been found [16, 17, 18, 19]: So far, effective versions of (3) can exist in the presence of holonomy modifications only if there is an additional function β multiplying the generator on the right-hand side:

$$[T[N_1], T[N_2]] = S[\beta(K)(N_1 \vec{\nabla} N_2 - N_2 \vec{\nabla} N_1)] . \quad (6)$$

The classical structure function (the inverse spatial metric) is therefore modified. Moreover, if using holonomies modifies the classical curvature dependence on some component K in the Hamiltonian by replacing the usual quadratic dependence by a function $f(K)$, then the effective algebra is closed for $\beta(K) = \frac{1}{2} d^2 f / dK^2$. In particular, if $f(K)$ has a maximum at large density or curvature (as it is often exploited in singularity-free bounce models of loop quantum cosmology), $\beta(K)$ is negative in this regime.

For instance, in spatially flat isotropic minisuperspace models, $K = \mathcal{H}$ can be taken as the Hubble parameter, and it appears as a connection component in holonomies (4) computed along curves in isotropic space. The boundedness of matrix elements of holonomies is often taken as a motivation to replace $K^2 = \mathcal{H}^2$ in the classical Friedmann equation by a function $f(K) = \sin^2(\ell K) / \ell^2$, with a length parameter ℓ that could be the Planck length. (With this modification, $K = \mathcal{H} + O(\ell^2)$ with quantum corrections.) Although not only the value of ℓ but also the precise functional form of $f(K)$ are subject to quantization ambiguities, which at present remain poorly controlled, the bounded nature of $f(K)$ is considered robust because it comes from the boundedness of holonomies for compact groups. One obtains a modified Friedmann equation [20]

$$\frac{\sin^2(\ell K)}{\ell^2} = \frac{8\pi G}{3} \rho \quad (7)$$

or, after computing the holonomy corrections by which K differs from \mathcal{H} [21, 22],

$$\mathcal{H}^2 = \frac{8\pi G}{3} \rho \left(1 - \frac{8}{3} \pi G \ell^2 \rho \right) . \quad (8)$$

(This equation is an exact effective equation only if the sole matter content is a free, massless scalar [23]. In all other cases, higher-derivative corrections appear which may rival effects from holonomy modifications [24]. So far, no sufficiently general form of higher time derivatives is known in these models.)

If one takes equation (7) at face value, it implies that the energy density must be bounded as long as ℓ is non-zero: $\rho \leq \rho_{\max} = 3/(8\pi G \ell^2)$. The version (8) of the equation then suggests that an extremum of the scale factor is reached at the maximum density ρ_{\max} which can be shown to be a minimum. Holonomy modifications of loop quantum cosmology therefore lead to bounce models. However, for $f(K)$ of the required form, one obtains (6) with a correction function $\beta(K) = \cos(2\ell K)$ which is negative at high density around the bounce.

When β is different from one, or even negative, the classical relation (3) is strongly modified. The drastic nature of this quantum correction can be understood by noting that (3) with the opposite sign is obtained for deformations of hypersurfaces in Euclidean space rather than Lorentzian space-time. Bounded curvature as a consequence of holonomy modifications in loop quantum gravity, as realized in bouncing solutions of (8), can therefore be obtained only at the expense of having signature change at large curvature [25, 26, 27]. No new assumption is required to arrive at this conclusion; one only combines effective theory with the general conditions that loop quantum gravity should have some semiclassical states (in a weak sense since the \hbar -order is not fixed) and that it provides an anomaly-free quantization with a closed algebra of symmetry generators.

For negative β , one obtains elliptic mode equations instead of hyperbolic ones, of the form

$$-\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} + \beta(K) \Delta u = S \quad (9)$$

with some source term S . In $\beta(K)$, K changes according to a background solution of (8). An initial-value problem is no longer well-posed; it needs to be replaced by a boundary-value problem in all four dimensions. If one were to use an initial-value problem even through the elliptic regime (at high curvature), one would obtain unstable solutions. Moreover, even a well-posed problem for the mixed-type differential equations one is dealing with in this context, found by Tricomi in the 1930s, generically leads to large mode amplitudes somewhere in a region of interest [28]. (Mixed-type partial differential equations appear also in models of transonic flow. Locally large amplitudes in this context are well-known from sonic booms.) While it is possible to use cosmological perturbation theory in order to derive the local form of mode equations (9) including their mixed-type nature [17], global solutions appear to be inconsistent with inhomogeneity being treated as a perturbation.

These results give rise to two final conclusions. First, as promised, there is a model in which quantum-field theory on a (bouncing) background gives drastically different results from a background-independent treatment which takes into account the symmetries of (quantum) space-time. The former approach would simply assume that the bouncing background has the standard space-time structure, and inhomogeneity could evolve through the bounce perturbatively. Such models have been developed, but even though they are sometimes thought of as models of quantum gravity (the title of [29], for instance, advertizes “a quantum gravity extension of the inflationary scenario” although it is based on classical assumptions about inhomogeneity in space-time), they assume classical properties of the background which are violated in currently known anomaly-free models with the same quantum corrections in the background dynamics (8).

The second conclusion is a sobering note on potential derivations of observational signatures of loop quantum gravity (while other approaches to quantum gravity not based on holonomy modifications may have a more positive outlook). The Planck regime in this theory appears to be too ambiguous to produce reliable pre-big bang information, and perturbative inhomogeneity is likely to be inadequate owing to instabilities induced by the very same quantum-gravity effects that have been exploited for singularity resolution. One might use the theory for computations of quantum corrections at sub-Planckian densities,

when the curvature parameter K is small enough for $\beta(K)$ to turn positive. The implication of loop modifications is then to replace the singular beginning of the expansion phase in the usual inflationary models by a non-singular beginning marked by the first emergence of time at the boundary of the Euclidean phase. Additional quantum corrections characteristic of the loop approach are small in this regime, but they could potentially be computed by effective methods and, once they are known in detail and with a classification of all possible quantum ambiguities within this theory, compared with observations. A key new effect, directly related to non-classical space-time structures, is related to modified propagation speeds of various modes implied by (9) for $0 < \beta \neq 1$. Importantly, scalar and tensor modes may have different speeds [30], so that characteristic modifications of the usual scalar-to-tensor ratio can result. The theory might have something to say about preferred initial states of an inflaton (see for instance [31]), but this issue is still being analyzed.

To conclude, surprising and perhaps useful effects have been found in models of loop quantum gravity. In scenarios of structure formation, the consequences are not always as they first appear in the simplest (homogeneous) models which by necessity are blind to the underlying quantum space-time structure. There remain several difficult questions to be addressed, for instance about anomalies, before one can make reliable predictions. Nevertheless, at a conceptual level the question of how loop quantum gravity could be combined with the inflationary scenario is producing several interesting lessons.

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